Quaternions, an overview.

The complex numbers are, numbers that have two components called real and imaginary that can often be interpreted as 2 dimensions. A complex number is normally written as $a + ib$ where $i^2 = -1$ and a and b are two real values quantities.
This idea can be extended to higher dimensions, but it turns out that 4 components have useful properties. These are called quaternions and are attributed to *Sir William Rowan Hamilton* who published a major analysis in 1844 called "On a Species of Imaginary Quantities Connected with a Theory of Quaternions" in the Proceedings of the Royal Irish Academ (2, pp 424-434).

Definition

In this discussion we will write a quaternion *"Q"* as

$$Q = r + a\,i + b\,j + c\,k$$

Where r is the real component and $a; b; c$ real values. This 4 $(r;a;b;c)$ might be considered to be a vector in the *4D quaternion space*.
When performing operations on complex numbers whenever one encounters i^2 then one knows that is equal to the simpler -1. There are similar but slightly more complicated relationships between $i;j;k$ in quaternion space. They are as follows:

$$i^2 = j^2 = k^2 = -1$$

$$i\,j = k \qquad j\,k = i \qquad k\,i = j$$

$$j\,i = -k \qquad k\,j = -i \qquad i\,k = -j$$

Note that the order in which $i;j;k$ appears in an expression is important. Also note that there is no linear relationship between $i;j;k$.

Addition

Addition (or subtraction) of two quaternions $Q_1 = r_1 + a_1\,i + b_1\,j + c_1\,k$ and $Q_2 = r_2 + a_2\,i + b_2\,j + c_2\,k$ is performed as follows.

$$Q_1 + Q_2 = r_1+r_2 + (a_1+a_2)\,i + (b_1+b_2)\,j + (c_1 + c_2)\,k$$

Congujate

The congujate of $Q = Q^* = r - a\,i - b\,j - c\,k$.

Multiplication

Multiplication of two quaternions is somewhat involved but follows directly from the relationships above.

$$\begin{aligned} Q_1 Q_2 = &[\, r_1 r_2 - a_1 a_2 - b_1 b_2 - c_1 c_2 \,] + \\ &[\, r_1 a_2 + a_1 r_2 + b_1 c_2 - c_1 b_2 \,]\, i + \\ &[\, r_1 b_2 + b_1 r_2 + c_1 a_2 - a_1 c_2 \,]\, j + \\ &[\, r_1 c_2 + c_1 r_2 + a_1 b_2 - b_1 a_2 \,]\, k \end{aligned}$$

Note that quaternion multiplication is not commutative, that is, $Q_1 Q_2$ is NOT the same as $Q_2 Q_1$

Length (modulus)

The length (magnitude) of a quaternion is the familiar coordinate length in 4 dimensional space.

$$|Q| = sqrt(\, Q\, Q^*\,)$$

where Q^* is the congujate (see later) which expands to

$$|Q| = sqrt(r^2 + a^2 + b^2 + c^2)$$

and

$$|Q_1\, Q_2| = |Q_1|\, |Q_2|$$

Inverse

The inverse of a quaternion Q^{-1} such that $Q\, Q^{-1} = 1$ is given by

$$Q^{-1} = \frac{r - ai - bj - ck}{|Q|^2}$$

The inverse of a normalised quaternion is simply the congujate, otherwise the magnitude of the inverse is $1/|Q|$. So the above expression normalises the quaternion and then scales by $1/|Q|$.

Division

Division of Q_1 by Q_2 is as follows

$$\frac{Q_1}{Q_2} = \frac{Q_1(2r_2 - Q_2)}{|Q_2|^2}$$

Exponential

If $m = sqrt(a^2 + b^2 + c^2)$ and v is the unit vector $(a,b,c) / m$ then the exponential of the quaternion Q is

$$exp(Q) = exp(r)\ [\ cos(m),\ v\ sin(m)\ ]$$

Polar Coordinates

The equivalent to polar coordinates in quaternion space are

$r = |Q|\ cos(theta1)$
$a = |Q|\ sin(theta1)\ cos(theta2)$
$b = |Q|\ sin(theta1)\ sin(theta2)\ cos(theta3)$
$c = |Q|\ sin(theta1)\ sin(theta2)\ sin(theta3)$

theta1 is known as the amplitude of the quaternion, theta2 and theta3 are the latitude (or co-latitude) and longitude respectively. The representative point of a quaternion is the normalised vector *(a,b,c)*, that is, where *(a,b,c)* intersects the unit sphere centered at the origin.

Rotation of a vector about another vector

To rotate a *3D* vector *"p"* by angle theta about a (unit) axis *"r"* one forms the quaternion

$$Q_1 = (0, p_x, p_y, p_z)$$

and the rotation quaternion

$$Q_2 = (cos(theta/2),\ r_x\ sin(theta/2),\ r_y\ sin(theta/2),\ r_z\ sin(theta/2)).$$

The rotated vector is the last three components of the quaternion

$$Q_3 = Q_2\, Q_1\, Q_2^*$$

It is easy to see that rotation in the opposite direction *(-theta)* can be achieved by reversing the order of the multiplication.

$$Q_3 = Q_2^*\, Q_1\, Q_2$$

Note also that the quaternian Q_2 is of unit magnitude, and needs to be in order to be a valid rotation.

Converting a quaternion to a matrix

Given a quaternion rotation the corresponding *3x3* rotation matrix *M* is given by

$$M = \begin{pmatrix} 1-2b^2-2c^2 & 2ab-2rc & 2ac+2rb \\ 2ab+2rc & 1-2a^2-2c^2 & 2bc-2rc \\ 2ac-2rb & 2bc+2rc & 1-2a^2-2b^2 \end{pmatrix}$$

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